Safe Learning-Based Control of Stochastic Jump Linear Systems: A Distributionally Robust Approach
Mathijs Schuurmans†, Pantelis Sopasakis†, Panagiotis Patrinos† | †KU Leuven, Belgium, ‡Queen’s University Belfast, Northern Ireland

Background and motivation

Safe learning-based control
- Control of stochastic systems requires knowledge of underlying probability distributions
- In practice, distributions are unknown
- Distributionally robust approach [1, 2]: control assuming worst-case distribution in ambiguity set \( \mathcal{A} \)

Gather data \( \rightarrow \) Ambiguity decreases \( \rightarrow \) Safely reduce conservativeness

Robust \( \rightarrow \) Stochastic

Applications
Rigorous statistical guarantees \( \rightarrow \) safety-critical applications e.g., autonomous driving, robotics, ... (physical interaction with humans)

Problem statement
We aim to stabilize a linear system
\[
x_{t+1} = A(x_t)w_t + B(w_t)u_t,
\]
where random variables \( w_t \in \mathcal{W} := \{1, 2, \ldots, k\} \), specify the operation mode \( A(i) = A_i, B(i) = B_i \) at time \( t \), and \( P : 2^\mathcal{W} \rightarrow \mathbb{R} \), with \( P[w = i] = P[\{i\}] = p_i \), is an unknown probability measure.

Challenge: Mean-square (MS) stability conditions depend on true distribution \( p \) [3].

Goal: Mean-square stability in probability
For a given confidence level \( 1 - \alpha \in (0, 1) \), compute a linear state feedback gain \( K \), which renders (1) MS stable with probability at least \( 1 - \alpha \).

Proposed approach
- Estimate distribution \( \hat{p} \) based on \( N \) i.i.d. samples \( \{w_i\}_{i=1}^N \)
- Ensure MS stability for all \( p \in \mathcal{A}(\hat{p}) := \{ p \in \Delta_k \mid \| p - \hat{p} \|_1 \leq \tau \} \)

Subproblems
1. Compute \( \min r \) such that,
   \[
P( p \in \mathcal{A}(\hat{p})) \geq 1 - \alpha
   \]
2. Efficiently compute \( K \) that is MS stabilizing for all \( p \in \mathcal{A}(\hat{p}) \)

I Bounding the ambiguity

PAC-type confidence bounds for the empirical probability distribution estimate:
\[
\text{if } r \leq \min \{\epsilon_{\text{DKW}}, \epsilon_k\} \Rightarrow \text{(2) holds}
\]

Dvoretzky-Kiefer-Wolfowitz
\[
r = \epsilon_{\text{DKW}}(\alpha, k, N) = 2k \sqrt{\frac{\ln \alpha}{N}}
\]
\[
r = \epsilon_k(\alpha, k, N) = O\left(\frac{k}{\sqrt{N}}\right)
\]

McDiarmid + min-max loss bounds [4]
\[
r = \max_{\alpha, k} \epsilon_k(\alpha, k, N) = O\left(\sqrt{\frac{k}{N}}\right)
\]

II Efficient computation of the feedback gain

Distributedly Robust Lyapunov-type stability condition:
\[
\exists \mathcal{P} > 0: \max _{x \in \mathcal{R}} \sum_{i=1}^{k} p_i^t V(\mathcal{A}x) \leq V(x) - \ell(x, Kx),
\]
with \( \mathcal{A} = A + BK, V(x) = x^T P x, \) and \( \ell(x, u) = x^T Q x + w^T R u, \) with \( Q > 0, R \succeq 0. \)

Method 1 – Vertex enumeration
\[
\ell_c: \text{ambiguity set is polytopic: } \mathcal{A}(\hat{p}) = \text{conv}(P(\hat{p}))_{\mathcal{F}_1},
\]
\[
\Rightarrow \text{ Reduced to finite number of LMI conditions.}
\]

References

Future work
- Extend to optimal control setting and nonlinear dynamics
- Relax i.i.d. assumption \( \rightarrow \) Markov jump systems

Experimental results

Computational cost
- Rapid growth of vertex count \( \rightarrow \) vertex enumeration applicable for very low dimensions (\( k \leq 7 \)) of the sample space \( \mathcal{W} \).
- Computation of the vertices of the ambiguity set alone is more time-consuming than solving the complete reformulated problem.

Sample complexity

Set-up
1. Given a closed-loop system
\[
x_{t+1} = (A(w_t) + KB(w_t))x_t
\]
with a \( p \)-MS controller \( K \) (stochastic approach).
2. Define distributional stability region \( \mathcal{D} := \{p \mid (4) \mbox{ p-MSS}\}
3. Compute \( \max_r, \text{s.t. } \mathcal{A}(\hat{p}) \subseteq \mathcal{D} \Rightarrow P((4) \mbox{ p-MSS}) \geq (1 - \alpha)
\]

For Bernoulli system in [5], \( \mathcal{A}(\hat{p}) \subseteq \mathcal{S} \) is easy to test.

For a given confidence level, the distributionally robust approach provides a stabilizing controller using several orders of magnitude less data.