# Safe Learning-Based Control of Stochastic Jump Linear Systems: A Distributionally Robust Approach

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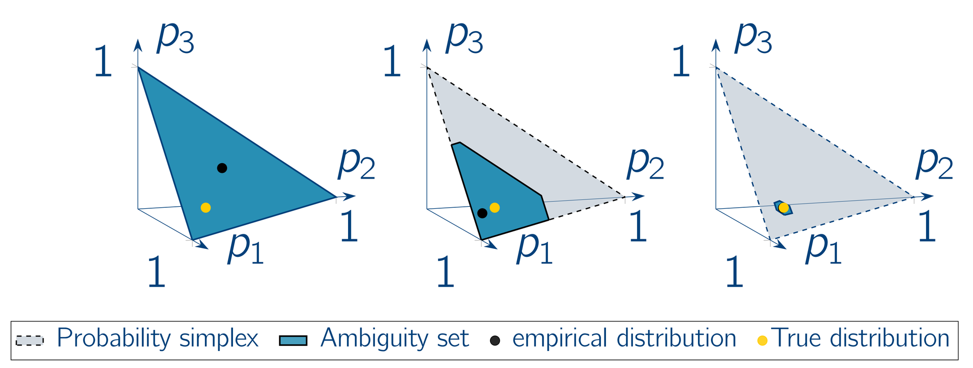
## Background and motivation

### Safe learning-based control

- Control of stochastic systems requires knowledge of underlying **probability distributions**
- In practice: distributions are **unknown**
- Distributionally robust approach [1, 2]: control assuming **worst-case distribution in ambiguity set $\mathcal{A}$**

Gather data → Ambiguity decreases → Safely reduce conservativeness




### Applications

Rigorous statistical guarantees → **safety-critical** applications e.g., autonomous driving, robotics, ... (physical interaction with humans)

## Problem statement

We aim to **stabilize** a linear system

$$x_{t+1} = A(w_t)x_t + B(w_t)u_t, \tag{1}$$

where random variables $w_t \in \mathcal{W} := \{1, 2, \ldots, k\}$, specify the operation mode ($A(i) = A_i$, $B(i) = B_i$) at time $t$, and $\mathrm{P} : 2^{\mathcal{W}} \to \mathrm{IR}$, with $\mathrm{P}[w = i] = \mathrm{P}[\{i\}] = p_i$ is an **unknown** probability measure.

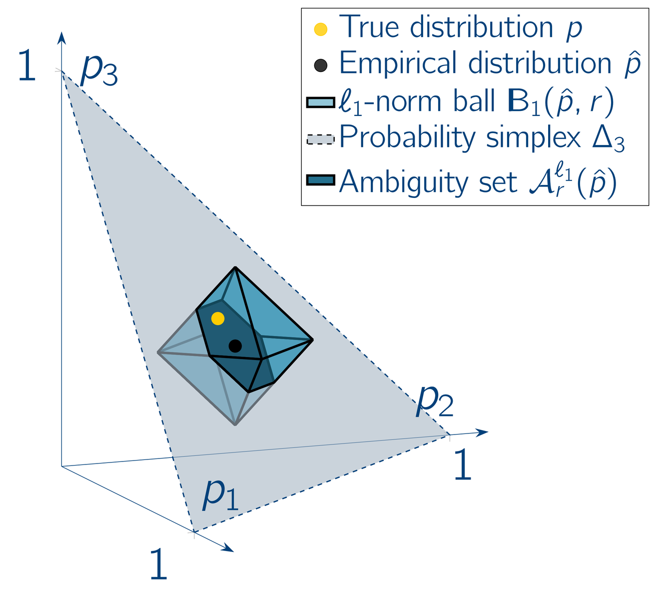
**Challenge:** Mean-square (MS) stability conditions depend on true distribution $p$ [3].

**Goal: Mean-square stability in probability**

For a given confidence level $1-\alpha \in (0, 1)$, compute a **linear state feedback gain** $K$, which renders (1) MS stable with probability at least $1 - \alpha$.

## Proposed approach

- **Estimate distribution** $\hat{p}$ based on $N$ i.i.d. samples $\{w_i\}_{i=1}^{N}$
- **Ensure MS stability** for all $p \in \mathcal{A}_r^{\ell_1}(\hat{p}) := \{p \in \Delta_k \mid \|p - \hat{p}\|_1 \leq r\}$




**Subproblems**

I Compute **min** $r$ such that,

$$\mathrm{P}[p \in \mathcal{A}_r^{\ell_1}(\hat{p})] \geq 1 - \alpha \tag{2}$$

II Efficiently compute $K$ that is MS stabilizing for all $p \in \mathcal{A}_r^{\ell_1}(\hat{p})$

## I Bounding the ambiguity

PAC-type confidence bounds for the empirical probability distribution estimate:

if $r \leq \mathbf{min}\{r_{\mathrm{DKW}}, r_{\mathrm{M}}\} \Rightarrow$ (2) holds

### Dvoretzky-Kiefer-Wolfowitz

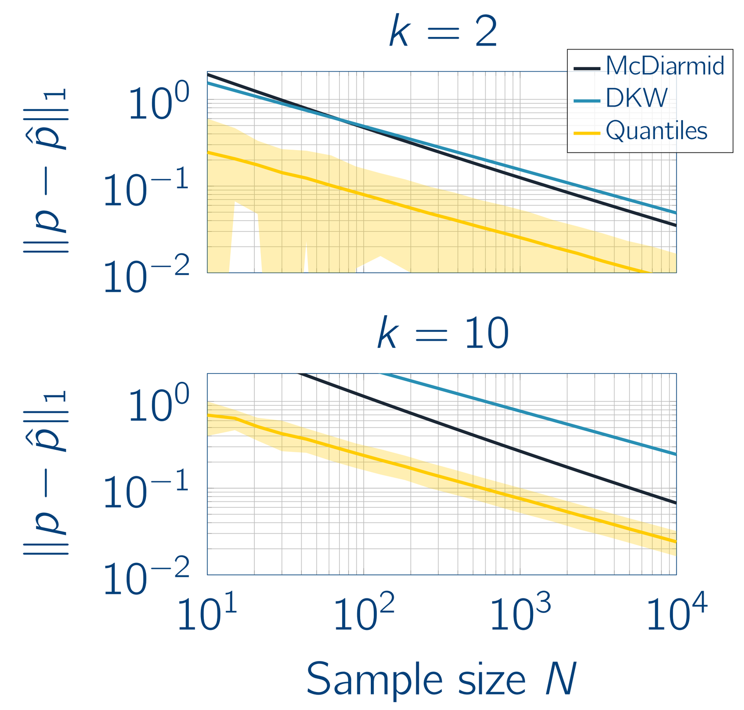
$$r = r_{\mathrm{DKW}}(\alpha, k, N) := 2k\sqrt{\frac{\ln {}^{2}\!/\!_{\alpha}}{2N}}$$

$$\Rightarrow r_{\mathrm{DKW}}(\,\cdot\,, k, N) = \mathcal{O}\left(\tfrac{k}{\sqrt{N}}\right)$$

### McDiarmid + min-max loss bounds [4]

$$r = r_{\mathrm{M}}(\alpha, k, N) := \sqrt{\frac{2\ln(1/\alpha)}{N}} + \sqrt{\frac{2(k-1)}{\pi N}} + \frac{4k^{1/2}(k-1)^{1/4}}{N^{3/4}}$$

$$\Rightarrow r_{\mathrm{M}}(\,\cdot\,, k, N) = \mathcal{O}\left(\frac{k^{3/4}}{\sqrt{N}}\right)$$




## II Efficient computation of the feedback gain

Distributionally Robust Lyapunov-type stability condition:

$$\exists P \succ 0 : \max_{p \in \mathcal{A}} \sum_{i=1}^{k} p_i V(\bar{A}_i x) \leq V(x) - \ell(x, Kx), \tag{3}$$

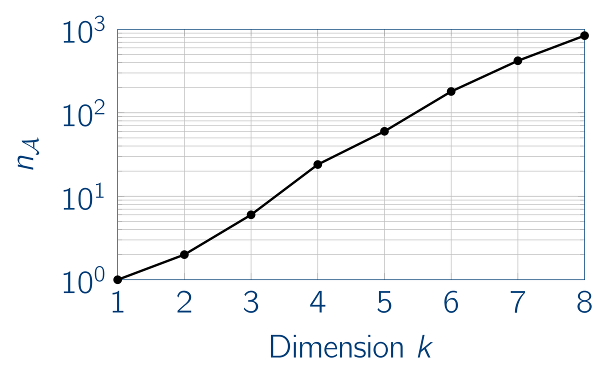
with $\bar{A}_i = A_i + B_i K$, $V(x) = x^\top P x$, and $\ell(x, u) = x^\top Q x + u^\top R u$, with $Q \succ 0$, $R \succeq 0$.

**Method 1 – Vertex enumeration**

$\ell_1$-ambiguity set is **polytopic**: $\mathcal{A}_r^{\ell_1}(\hat{p}) = \mathbf{conv}\{p^{(i)}\}_{i=1}^{n_{\mathcal{A}}}$.

$$(3) \Leftrightarrow \max_{l \in \mathbb{N}_{[1, n_{\mathcal{A}}]}} \sum_{i=1}^{k} p_i^{(l)} V(\bar{A}_i x) \leq V(x) - \ell(x, Kx)$$

→ Reduced to finite number of LMI conditions.




**Drawback:** computational cost of 1) computing vertices; and 2) solving LMI for all vertices.

**Method 2 – Reformulation**

Reformulation based on 3 key observations:

1 $\max_{p \in \mathcal{A}} \sum_{i=1}^{k} p_i V(\bar{A}_i x) = \sigma_{\mathcal{A}}(V(\bar{A}_i x))$

2 $\mathcal{A}_r^{\ell_1}(\hat{p}) = \Delta_k \cap \mathrm{B}_1(\hat{p}, r) = \Delta_k \cap C$ ⟶

3 $\sigma_{\Delta_k \cap C}(v) = (\sigma_{\Delta_k} \,\square\, \sigma_C)(v)$

Infimal convolution: $(f \,\square\, g)(v) = \mathbf{inf}_z\, f(v - z) + g(z)$

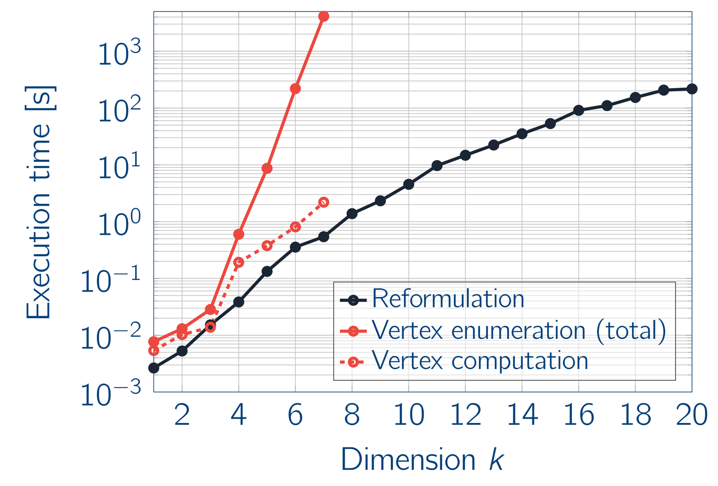
Easily computable support function of elementary sets

$\sigma_{\Delta_k}(v) = \mathbf{max}\{v_1, \ldots, v_k\}$
$\sigma_C(v) = r\|v\|_\infty + v^\top \hat{p}$

Close approximation leads to $2K^2$ LMIs.

## Experimental results

### Computational cost




- Rapid growth of vertex count → vertex enumeration applicable for very low dimensions ($k \leq 7$) of the sample space $\mathcal{W}$.
- Computation of the vertices of the ambiguity set alone is more time-consuming than solving the complete reformulated problem.

### Sample complexity

**Set-up**

1 Given a closed-loop system

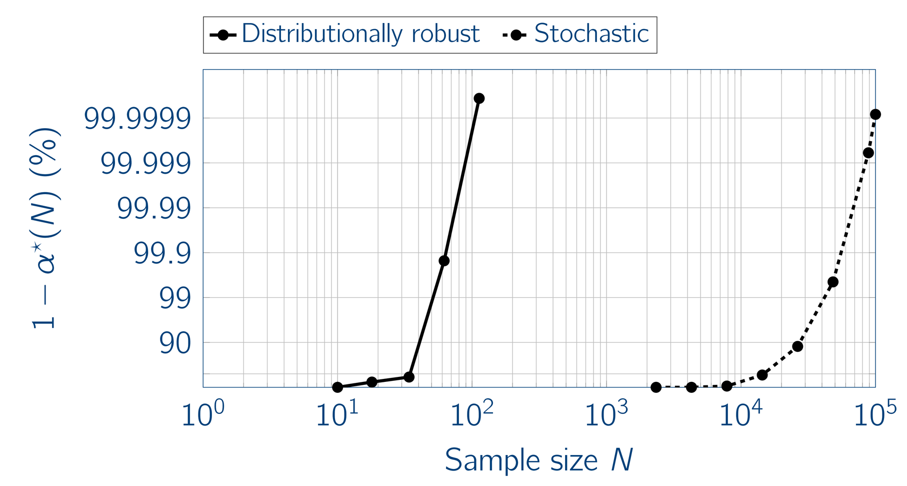
$$x_{t+1} = \Big(A(w_t) + KB(w_t)\Big)x_t \tag{4}$$

with a $\hat{p}$-MSS controller $K$ *(stochastic approach)*

2 Define **distributional stability region** $\mathcal{S} := \{p' \mid (4)\ p'\text{-MSS}\}$

3 Compute **max** $r$, s.t. $\mathcal{A}_r^{\ell_1}(\hat{p}) \subseteq \mathcal{S} \Rightarrow \mathrm{P}((4)\ p\text{-MSS}) \geq (1 - \alpha)$

For Bernoulli system in [5], $\mathcal{A}_r^{\ell_1}(\hat{p}) \subseteq \mathcal{S}$ is easy to test.




For a **given confidence level**, the distributionally robust approach provides a stabilizing controller using several orders of magnitude **less data**.

## Future work

- Extend to **optimal control** setting and nonlinear dynamics
- Relax i.i.d. assumption → Markov jump systems

### References

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