Data-Driven Model Predictive Control with Stability and Robustness Guarantees

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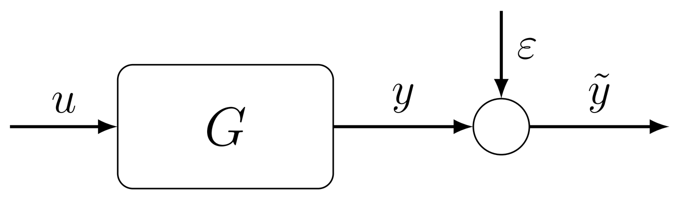
Setting - Unknown LTI System



- Control of a linear time-invariant (LTI) system G of order n, directly from measured data $\{u_k^d, y_k^d\}_{k=0}^{N-1}$, without prior system identification.
- **Goal:** Tracking of a setpoint (u^s, y^s).
- **Method:** Data-driven model predictive control (MPC).
- Theoretical guarantees with noisy output measurements $\rightarrow$ *robust* data-driven MPC.

Trajectory-based System Representation

For a sequence $\{u_k\}_{k=0}^{N-1}$ define the Hankel matrix

$$H_L(u) := \begin{bmatrix} u_0 & u_1 & \dots & u_{N-L} \\ u_1 & u_2 & \dots & u_{N-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{L-1} & u_L & \dots & u_{N-1} \end{bmatrix}.$$

Definition: A signal $\{u_k\}_{k=0}^{N-1}$ with $u_k \in \mathbb{R}^m$ is persistently exciting of order L if $\text{rank}(H_L(u)) = mL$.

Theorem [1, 2]

Suppose $\{u_k^d, y_k^d\}_{k=0}^{N-1}$ is a trajectory of an LTI system G, where u is persistently exciting of order $L+n$. **Then**, $\{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1}$ is a trajectory of G **if and only if** there exists $\alpha \in \mathbb{R}^{N-L+1}$ such that

$$\begin{bmatrix} H_L(u^d) \\ H_L(y^d) \end{bmatrix} \alpha = \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}. \quad (1)$$

Implications

- Time-shifts of measured data span all other trajectories.
- Model-free, data-driven system parametrization.
- Only requirements: persistence of excitation & upper bound on model order.
- Has been employed to verify dissipativity [3] and to design stabilizing controllers [4].

Robust Data-Driven Model Predictive Control

Noisy output data $\tilde{y}_k = y_k + \varepsilon_k$ with $\|\varepsilon_k\|_\infty \leq \bar{\varepsilon}$ in a) initial data $\tilde{y}^d$ and b) online data $\tilde{y}_t$.
MPC Problem: Given the past n input-output measurements, solve

$$J_L^*(u_{[t-n,t-1]}, \tilde{y}_{[t-n,t-1]}) = \min_{\substack{\alpha(t),\sigma(t) \\ \bar{u}(t),\bar{y}(t)}} \sum_{k=0}^{L-1} \ell(\bar{u}_k(t), \bar{y}_k(t)) + \lambda_\alpha \bar{\varepsilon}^2 \|\alpha(t)\|_2^2 + \lambda_\sigma \|\sigma(t)\|_2^2$$

$$s.t. \quad \begin{bmatrix} \bar{u}(t) \\ \bar{y}(t) + \sigma(t) \end{bmatrix} = \begin{bmatrix} H_{L+n}(u^d) \\ H_{L+n}(\tilde{y}^d) \end{bmatrix} \alpha(t), \quad (2a)$$

$$\begin{bmatrix} \bar{u}_{[-n,-1]}(t) \\ \bar{y}_{[-n,-1]}(t) \end{bmatrix} = \begin{bmatrix} u_{[t-n,t-1]} \\ \tilde{y}_{[t-n,t-1]} \end{bmatrix}, \quad (2b)$$

$$\begin{bmatrix} \bar{u}_{[L-n,L-1]}(t) \\ \bar{y}_{[L-n,L-1]}(t) \end{bmatrix} = \begin{bmatrix} u_n^s \\ y_n^s \end{bmatrix}, \quad (2c)$$

$$\bar{u}_k(t) \in \mathbb{U}, \quad k \in \mathbb{I}_{[0,L-1]}. \quad (2d)$$

Ingredients:

- Prediction / system dynamics (2a), initial conditions (2b), terminal constraints (2c), input constraints (2d), no output constraints $\rightarrow$ future research.
- Slack variable σ: account for noise in (2a) and (implicitly) in (2b), regularized in the cost.
- Stage cost $\ell(u, y) = \|u - u^s\|_R^2 + \|y - y^s\|_Q^2$.
- Regularization of $\alpha \rightarrow$ improves signal-to-noise-ratio.
- Same scheme without (2c) suggested in [5] $\rightarrow$ no terminal constraints $\rightarrow$ no guarantees on recursive feasibility and stability.
- **Goal:** Prove recursive feasibility and stability of the closed loop.

Algorithm: Multi-Step Robust Data-Driven MPC Scheme

1. At time t, take the past n measurements $u_{[t-n,t-1]}$, $\tilde{y}_{[t-n,t-1]}$ and solve (2).
2. Apply the optimal input sequence $\bar{u}^*_{[0,n-1]}(t)$ over the next n time steps.
3. Set $t = t + n$ and go back to 1).

Theoretical Results

Define $H_{ux} = \begin{bmatrix} H_{L+n}(u^d_{[0,N-1]}) \\ H_1\left(x^d_{[0,N-L-n]}\right) \end{bmatrix}$, $c_{pe} = \left\|H_{ux}^\dagger\right\|_2^2$, and consider $(u^s, y^s) = (0, 0)$.

Proposition (recursive feasibility)

Suppose u^d is persistently exciting of order $L + 2n$ and $L \geq 2n$.
Then, there exists $\bar{\varepsilon}_0$ such that, for any $\bar{\varepsilon} \leq \bar{\varepsilon}_0$, if the robust MPC problem (2) is feasible at initial time $t = 0$, then the n-step MPC scheme is feasible at any $t \in \mathbb{N}$.

Theorem (practical stability)

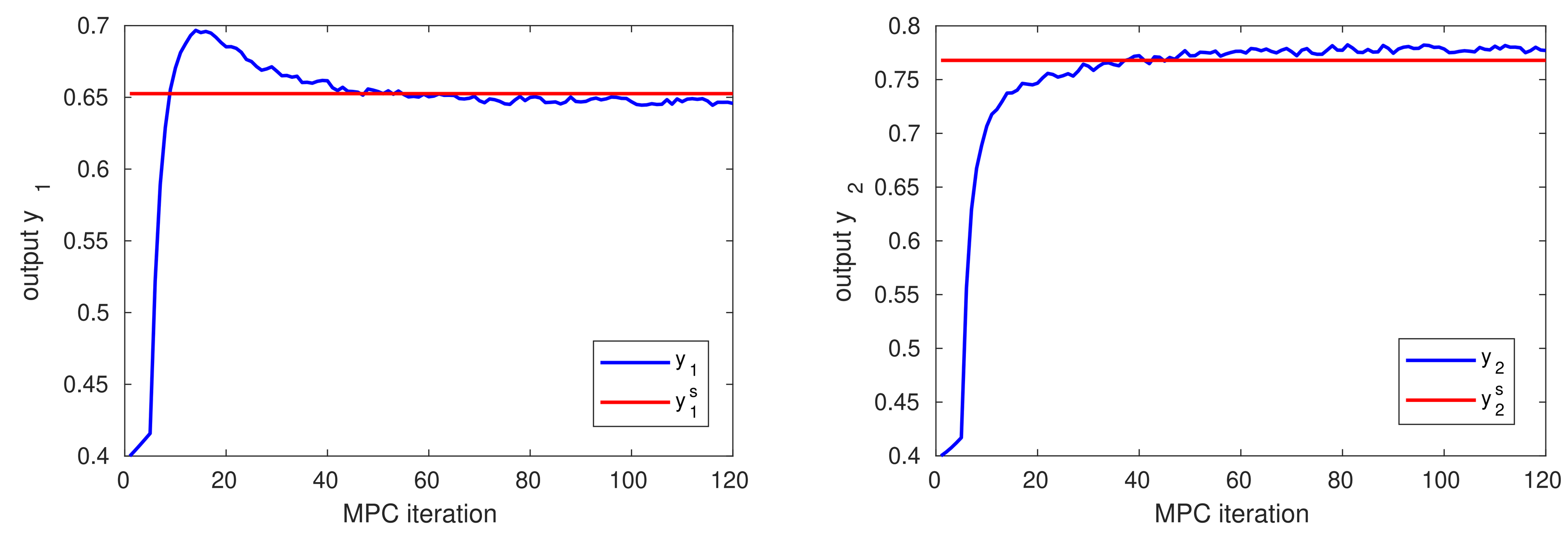
Suppose u^d is persistently exciting of order $L + 2n$ and $L \geq 2n$.
For any $V_{ROA} > 0$, there exist suitable $\lambda_\alpha, \lambda_\sigma$ as well as bounds $\bar{\varepsilon}_0, \bar{c}_{pe}$ such that for any $\bar{\varepsilon} \leq \bar{\varepsilon}_0$, $c_{pe}\bar{\varepsilon}^2 \leq \bar{c}_{pe}$, the origin is *practically exponentially stable* w.r.t. $\bar{\varepsilon}$ with region of attraction $\{x_0 \in \mathbb{R}^n \mid V(x_0) \leq V_{ROA}\}$.

- $V = J_L^* + W$ (W positive definite) is a Lyapunov function for the closed loop of the multi-step MPC scheme.
- The closed loop converges to a set, whose size depends on the noise bound $\bar{\varepsilon}$.
- Same properties hold locally for a 1-step scheme.
- $\lambda_\alpha, \lambda_\sigma$ must be large enough for stability, but not too large for robustness.
- $c_{pe}\bar{\varepsilon}^2 \leq \bar{c}_{pe}$ is a quantitative "persistence-of-excitation-to-noise"-ratio and determines the size of V_{ROA}, i.e., the region of attraction.

Example

Realistic four tank system [6]:

- Stable, 4 states, 2 inputs, 2 outputs, noisy measurements with $N = 400$, $\bar{\varepsilon} = 0.005$.
- No model knowledge.
- Design parameters: $L = 30,\ Q = I,\ R = 5 \cdot 10^{-4} I,\ \lambda_\sigma = 1000,\ \lambda_\alpha \bar{\varepsilon}^2 = 0.1$.
- **Goal:** Tracking of a desired setpoint (u^s, y^s).



(a) Closed-loop output y_1 (b) Closed-loop output y_2

Confirms theoretical results:

- The proposed scheme tracks the desired setpoint & the tracking error increases with $\bar{\varepsilon}$.
- Inadequate tuning of $\lambda_\alpha, \lambda_\sigma \rightarrow$ unstable closed loop.
- The scheme without terminal constraints [5] destabilizes the system.

Conclusion

- First results on data-driven (model-free) MPC with stability and robustness *guarantees*.
- Connection between system & design parameters and closed-loop region of attraction.
- Theoretical results confirmed in practical example.

References

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